Siberian Mathematical Journal, Vol. 42, *No.* 1, *pp.* 181–189, 2001

ON THE L_w^2-SOLUTIONS OF GENERAL SECOND-ORDER NONSYMMETRIC DIFFERENTIAL EQUATIONS

I. Sobhy El-sayed, N. Faried, and G. M. Attia UDC 517.91

1. Introduction. Amos proved in [1] that all solutions of the second-order ordinary differential equation $M[y] = \lambda wy$ ($\lambda \in \mathbb{C}$) are in $L_w^2(a,\infty)$, when M is a second-order symmetric ordinary differential expression in the form $M[f] = -(pf')' + qf$ on $[a,\infty)$ ($' \equiv \frac{d}{dx}$), under sufficient conditions on the coefficients p and q. The case in which not all solutions are in $L_w^2(a,\infty)$ was considered by Atkinson and Evans in [2, Theorem 1]. Here we are concerned with the $L_w^2(a,\infty)$-solutions of the general second-order nonsymmetric differential equations $M[f] = \lambda wf$ and $M^+[g] = \bar{\lambda} wg$, where $M[.]$ is defined by

$$M[f] = -(p(f' - rf))' + up(f' - rf) + qf \text{ on } [a,b) \tag{1.1}$$

for a suitable complex-valued function f and its formal adjoint is

$$M^+[g] = -(p(g' + uf))' + rp(g' + uf) + qg \text{ on } [a,b). \tag{1.2}$$

The coefficients p, r, u, and q are complex-valued functions Lebesgue measurable on the interval $[a,b)$ of the real axis, $-\infty < a < b \le \infty$, and satisfy the following conditions:

$$p(x) \neq 0 \text{ for almost all } x \in [a,b), \quad \frac{1}{p}, r, u, q \in L_{\text{loc}}(a,b), \tag{1.3}$$

where $L_{\text{loc}}(a,b)$ denotes the space of all complex-valued functions integrable over every compact subinterval of $[a,b)$.

Our objective in this paper is to extend the results in [1] and [2] to a general second-order nonsymmetric differential expression M under sufficient conditions on the complex-valued coefficients of M.

2. Preliminaries. We denote by L_{loc} and AC_{loc} Lebesgue integration and absolute continuity; the subscript loc means "local" and restricts properties to compact subintervals of $\mathbb{R}$; and $\bar{z}$ denotes the conjugate of $z \in \mathbb{C}$.

Let w be a weight function satisfying:

$$w : [a,b) \to \mathbb{R}, \quad w(x) > 0 \quad (\text{for almost all } x \in [a,b)). \tag{2.1}$$

The equation

$$M[y] = \lambda wy \text{ on } [a,b) \quad (\lambda \in \mathbb{C}) \tag{2.2}$$

is said to be regular at the left endpoint a if

$$a > -\infty, \quad r, u, q, w \in L_{\text{loc}}[a,b). \tag{2.3}$$

Otherwise (2.2) is called *singular* at a. A similar terminology is used for the endpoint b. If $M[.]$ is regular at the endpoints a and b then we say that $M[.]$ is regular *on the interval* $[a,b]$; see [3–6].

We will consider the case when a is regular for (2.2) while b is singular. The endpoint a is regular for the equation (2.2) if and only if it is regular for the equation,

$$M^+[g] = \bar{\lambda} wg \text{ on } [a,b) \quad (\lambda \in \mathbb{C}). \tag{2.4}$$

Egypt. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 42, No. 1, pp. 202–211, January–February, 2001. Original article submitted March 10, 1999.

Let $H = L^2_w[a,b)$ denote the Hilbert function space of equivalence classes of Lebesgue measurable functions with the norm $\|f\|_w := \int_a^b w|f|^2 < \infty$ whose inner product is defined by

$$(f,g)_w := \int\limits_a^b f(x)\overline{g(x)}w(x)\,dx. \tag{2.5}$$

The domain and range of a linear operator T acting in a Hilbert space H are denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ is the null space of T. The nullity of T, written $\operatorname{nul}(T)$, is the dimension of $N(T)$ and the deficiency of T, written $\operatorname{def}(T)$, is the codimension of $R(T)$ in H; thus, if T is densely defined and $R(T)$ is closed then $\operatorname{def}(T) = \operatorname{nul}(T^*)$. The Fredholm domain of T is (in the notation of [3]) the open subset $\triangle_3(T)$ such that $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $\operatorname{ind}(T - \lambda I) = \operatorname{nul}(T - \lambda I) - \operatorname{def}(T - \lambda I)$ which is defined for $\lambda \in \triangle_3(T)$.

The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that $\|(A - \lambda I)x\| \geq K(\lambda)\|x\|$ for all $x \in D(A)$. By the Closed Graph Theorem, this is equivalent to the fact that $\operatorname{nul}(T - \lambda I) = 0$ and $R(T - \lambda I)$ is closed.

The joint field of regularity $\Pi(A,B)$ of A and B is the set of $\lambda \in \mathbb{C}$ such that $\lambda \in \Pi(A)$, $\bar\lambda \in \Pi(B)$, and both $\operatorname{def}(A - \lambda I)$ and $\operatorname{def}(B - \bar\lambda I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A,B) \neq \varnothing$.

For a general differential expression M considered here, the minimal operator is no longer symmetric. However, the minimal operators $T_0(M)$ and $T_0(M^+)$ generated by M and M^+ do form an adjoint pair in the sense that $T_0(M) \subset [T_0(M^+)]^*$ and $T_0(M^+) \subset [T_0(M)]^*$ or equivalently $(T_0(M)f,g) = (f, T_0(M^+)g)$ for $f \in D_0(M)$ and $g \in D_0(M^+)$, where $(.,.)_{L^2_w(a,b)}$ is the $L^2_w\{a,b)$ inner product defined in (2.5), and $D_0(M)$ and $D_0(M^+)$ are the domains of $T_0(M)$ and $T_0(M^+)$ respectively (see (2.6) below). Set

$$D(M) := \{f \in L^2_w(a,b), f^{[s-1]} \in AC_{\text{loc}}[a,b),\ s = 1,2, \text{ and } w^{-1}M[f] \in L^2_w(a,b)\},$$

$$D(M^+) := \{g \in L^2_w(a,b), g^{[s-1]} \in AC_{\text{loc}}[a,b),\ s = 1,2, \text{ and } w^{-1}M^+[g] \in L^2_w(a,b)\},$$

where $f^{[s-1]}$ and $g^{[s-1]}$ are the quasiderivatives defined by $f^{[0]} := f$, $f^{[1]} := p(f' - rf)$; $g^{[0]} := g$, $g^{[1]}_+ := p(g' + ug)$, and $AC_{loc}[a,b)$ denotes the set of functions absolutely continuous over every compact subinterval of $[a,b)$.

The subspaces $D(M)$ and $D(M^+)$ of $L^2_w(a,b)$ are the domains of the so-called maximal operators $T(M)$ and $T(M^+)$ defined by $T(M)f := w^{-1}M[f]$ $(f \in D(M))$ and $T(M^+)g := w^{-1}M^+[g]$ $(g \in D(M^+))$.

For a regular problem, the minimal operators $T_0(M)$ and $T_0(M^+)$ are the restrictions of $w^{-1}M[f]$ and $w^{-1}M^+[g]$ to the respective subspaces:

$$\begin{aligned} D_0(M) &:= \{f : f \in D(M),\ f^{[s-1]}(a) = f^{[s-1]}(b) = 0 \text{ for } s = 1,2\}, \\ D_0(M^+) &:= \{g : g \in D(M^+),\ g^{[s-1]}_+(a) = g^{[s-1]}_+(b) = 0 \text{ for } s = 1,2\}. \end{aligned} \tag{2.6}$$

The subspaces $D_0(M)$ and $D_0(M^+)$ are dense in $L^2_w(a,b)$, and $T_0(M)$ and $T_0(M^+)$ are closed operators (see [3, Theorem 3.10.5] and [4, Theorem 10]).

In a singular problem we first introduce the operators $T'_0(M)$ and $T'_0(M^+)$: $T'_0(M)$ is the restriction of $w^{-1}M[.]$ to the subspace $D'_0(M) := \{f : f \in D(M), \operatorname{supp} f \subset (a,b)\}$, and $T'_0(M^+_p)$ is defined similarly. These operators are densely-defined and closable in $L^2_w(a,b)$; and we define the minimal operators $T_0(M)$ and $T_0(M^+)$ to be their respective closures (cf. [7, Section 5]). We denote the domains of $T_0(M)$ and $T_0(M^+)$ by $D_0(M)$ and $D_0(M^+)$ respectively. It can be shown that

$$f \in D_0(M) \Rightarrow f^{[s-1]}(a) = 0\ (s = 1,2), \quad g \in D_0(M^+) \Rightarrow g^{[s-1]}_+(a) = 0\ (s = 1,2), \tag{2.7}$$